Schemes in Lean

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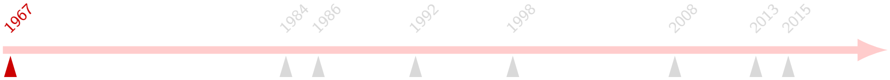
June 24, 2019

Formalising mathematics

Idea: Use a computer to check every step in a proof.

Why?

- There are many errors in published work.
- Some proofs are long and tedious to check 'by hand'.
- Build a database of all mathematical knowledge.



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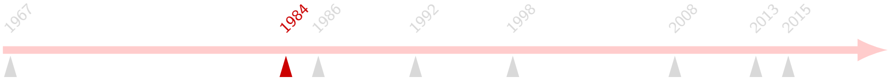

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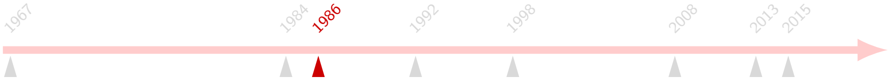

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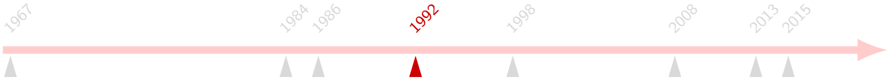

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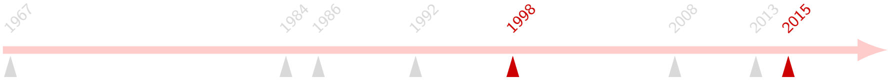

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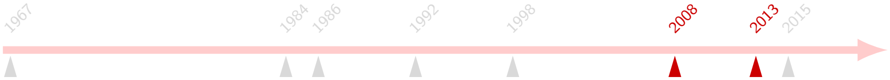

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Lean

Lean was launched in 2013 and is developed mainly by Leonardo de Moura at Microsoft Research. See `http://leanprover.github.io`.

Features:

- Based on the Calculus of Inductive Constructions.
  - Types: dependent types, inductive types, `Prop` and `Type u`.
  - Terms: types, constants, lambda abstractions and applications.
  - To give a proof of a statement `P : Prop` we build a term `p : P`.
- Supports powerful automation.
- Still at an early stage but promising standard library: mathlib.

(Xena project: Lean sessions every Thursday at the MLC)

Demo

Schemes

In 1960, the first volume of *Éléments de géométrie algébrique*, written by Alexander Grothendieck, is published.

Goal: Establish new foundations of algebraic geometry.

Approach: Define a *scheme* and build upon it.

Example

Suppose we want to study the polynomials $x = 0$ and $x^2 = 0$ over $\mathbb{C}$.

- Varieties: $Z(x) = Z(x^2) = \{0\}$.
- Schemes: $Spec(\mathbb{C}[x]/(x)) = \{(0)\}$ and $Spec(\mathbb{C}[x]/(x^2)) = \{(x)\}$.

Idea of an *affine scheme*:

- The set of functions on a space forms a ring.
- Start with a ring R.
- Define a topological space $Spec(R)$.
- Attach to it a *sheaf* of functions $\mathscr{O}_{Spec(R)}$ such that $\mathscr{O}_{Spec(R)}(Spec(R)) = R$ but do it in a way that it also makes sense to consider subsets of $Spec(R)$.

If we do it right, we have:

$$\{\text{Rings}\} \rightleftarrows \{\text{Affine schemes}\}$$
$$\text{Algebra} \rightleftarrows \text{Geometry}$$

This project

In March 2018, Kevin, Chris and Kenny defined a scheme in Lean.

Problems:

- Chaotic repository.
- Long and unnecessarily technical proofs.
- The definition was only mathematically equivalent.

Objective: start from scratch and do it well.

Canonical isomorphisms

In *Étale Cohomology*, a book about schemes by J. S. Milne we find:

> An injection is denoted by $\hookrightarrow$, a surjection by $\twoheadrightarrow$, an isomorphism by $\approx$, a quasi-isomorphism (or homotopy) by $\sim$, and a canonical isomorphism by $=$. The symbol $X \stackrel{df}{=} Y$ means X is defined to be Y, or that X equals Y by definition.

It brings up some questions:

- What is a canonical isomorphism?
- How does it look like in Lean?
- Is it the same as equality?

Canonical isomorphisms

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It brings up some questions:

- What is a canonical isomorphism? An 'obvious' isomorphism.
- How does it look like in Lean? Inverse maps in both directions.
- Is it the same as equality? No!

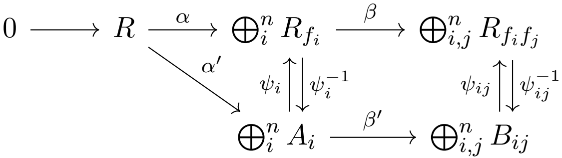
Definition

Suppose R is a ring and $f \in R$ then the *localisation of R away from f* is $R_f = \{(a, f^n) \mid a \in R, n \in \mathbb{N}\}/\sim$ where $(a, f^n) \sim (b, f^m)$ if and only if $f^k(af^m - bf^n) = 0$ for some k. We write $\frac{a}{f^n}$ to mean $[(a, f^n)]$.

We wanted to use a lemma about the sequence:

$$0 \longrightarrow R \xrightarrow{\alpha} \bigoplus_i^n R_{f_i} \xrightarrow{\beta} \bigoplus_{i,j}^n R_{f_i f_j}$$

But we didn't have R_{f_i} and $R_{f_i f_j}$. Instead, we had $A_i \cong R_{f_i}$ and $B_{ij} \cong R_{f_i f_j}$, so the situation was:



Solution:

- Neil Strickland came up with three axioms that characterise the class of rings isomorphic to R_{f_i} (respectively $R_{f_i f_j}$).
- Prove the lemma about the sequence for any type satisfying the three axioms.
- Show that A_i (respectively B_{ij}) satisfies the axioms.
- Avoid arguing about α', β', ψ_i, ψ_i^{-1}, ψ_{ij} and ψ_{ij}^{-1}.

Locally ringed spaces

The definition of a locally ringed space has three components:

1. A topological space X.
2. A sheaf of rings on X, denoted $\mathscr{O}_X$.
3. The property that, for all $x \in X$, the *stalk* $\mathscr{O}_{X,x}$ is a *local ring*.

The previous definition included 1 and 2. 3 followed by construction.

Theorem

The topological space $Spec(R)$ has the structure of a locally ringed space.

Over 7000 lines of code building up to the following.

Definition

A *scheme* is a locally ringed space $(X, \mathscr{O}_X)$ with an open cover $\{U_i\}$ such that for all i, $(U_i, \mathscr{O}_X|_{U_i})$ is isomorphic to $\left(Spec(R), \mathscr{O}_{Spec(R)}\right)$ for some ring R.

```
structure scheme (X : Type u) [topological_space X] :=
(carrier     : locally_ringed_space X)
(Haffinecov : ∃ (OC : covering.univ X), ∀ i,
   ∃ (R : Type v) [comm_ring R]
     (fpU : open_immersion_pullback (Spec R) carrier.O.F),
     fpU.range = OC.Uis i
   ∧ fpU.carrier ≅ structure_sheaf.presheaf R)
```

Conclusion and future work

We have given the first complete formal definition of a scheme and it is available in `https://github.com/ramonfmir/lean-scheme`.

Next steps:

- Kenny proved the correspondence between the category of rings and the category of affine schemes.
- Have an example of a projective scheme.
- Add schemes to mathlib!

Questions